A construction of representations for quantum groups: an example of $\mathcal{U}_q(\mathfrak{so}(5))$

Pavel Šťovíček

Department of Mathematics

Faculty of Nuclear Science, CTU

Trojanova 13, 120 00 Prague, Czech Republic

stovicek@kmdec.fjfi.cvut.cz

Abstract

A short description is given of a construction of representations for quantum groups. The method uses infinitesimal dressing transformation on quantum homogeneous spaces and is illustrated on an example of $\mathcal{U}_q(\mathfrak{so}(5))$.

1 Introduction

The purpose of this paper is to illustrate a construction of representations on an explicit example, namely the deformed enveloping algebra $\mathcal{U}_q(\mathfrak{so}(5))$. We are going to describe the construction as well, however, its detailed presentation will appear elsewhere. The basic ingredient is the infinitesimal dressing transformation on a quantum homogeneous space, in analogy with the celebrated method of orbits due to Kirillov and Kostant.

The construction generalizes and simplifies some results derived in the papers [1, 2, 3, 4, 5] and also [6, 7]. Let us mention just a few additional papers dealing also with constructions of representations of quantum groups and/or with quantum homogeneous spaces [8, 9, 10, 11], but taking a different point of view or applying other methods.

Concerning the deformation parameter, we assume that $q > 0$, $q \neq 1$. All fractional powers of q are supposed to be positive.

2 Construction

We assume that we are given a bialgebra $\mathcal{U}$ with the counit denoted by ε and the comultiplication denoted by Δ, and a unital algebra $\mathcal{C}$. Moreover, $\mathcal{C}$ is supposed to be a left $\mathcal{U}$-module with the action denoted by ξ, and fulfilling two conditions:

$$\xi_x \cdot 1 = \varepsilon(x)\,1, \quad \forall x \in \mathcal{U}, \tag{1}$$

$$\xi_x \cdot (fg) = (\xi_{x_{(1)}} \cdot f)(\xi_{x_{(2)}} \cdot g), \quad \forall x \in \mathcal{U},\ \forall f, g \in \mathcal{C}. \tag{2}$$

If convenient we shall write $\xi(x) \cdot f$ instead of $\xi_x \cdot f$. The second condition (2) is nothing but Leibniz rule. Here and everywhere in what follows we use Sweedler's notation: $\Delta x = x_{(1)} \otimes x_{(2)}$.

Proposition 1 *Suppose that a linear mapping $\varphi : \mathcal{U} \to \mathcal{C}$ satisfies $\varphi(1) = 1$ and*

$$\varphi(xy) = (\xi_{x_{(1)}} \cdot \varphi(y))\varphi(x_{(2)}), \quad \forall x, y \in \mathcal{U}. \tag{3}$$

Then the prescription

$$x \cdot f := (\xi_{x_{(1)}} \cdot f)\,\varphi(x_{(2)}), \quad \forall x \in \mathcal{U},\ \forall f \in \mathcal{C}, \tag{4}$$

defines a left $\mathcal{U}$-module structure on $\mathcal{C}$ and it holds

$$x \cdot (fg) = (\xi_{x_{(1)}} \cdot f)(x_{(2)} \cdot g), \quad \forall x \in \mathcal{U},\ \forall f, g \in \mathcal{C}. \tag{5}$$

Particularly,

$$\varphi(x) = x \cdot 1, \quad \forall x \in \mathcal{U}. \tag{6}$$

Conversely, suppose that $\mathcal{U} \otimes \mathcal{C} \to \mathcal{C} : x \otimes f \mapsto x \cdot f$ is a left $\mathcal{U}$-module structure on $\mathcal{C}$ such that the rule (5) is satisfied. Then the linear mapping $\varphi : \mathcal{U} \to \mathcal{C}$ defined by the equality (6) fulfills (3), and consequently the prescription (4) holds true.

Let us suppose, as usual, that $\mathcal{U}$ is generated as an algebra by a set of generators $\mathcal{M} \subset \mathcal{U}$. Let $\mathcal{F}$ be the free algebra generated by $\mathcal{M}$. Thus $\mathcal{U}$ is identified with a quotient $\mathcal{F}/\langle\mathcal{R}\rangle$ where $\langle\mathcal{R}\rangle$ is the ideal generated by a set of defining relations $\mathcal{R} \subset \mathcal{F}$. Let π be the factor morphism, $\pi : \mathcal{F} \to \mathcal{U}$. We set $\tilde{\varepsilon} := \varepsilon \circ \pi$ and

$$\tilde{\xi}_x \cdot f := \xi_{\pi(x)} \cdot f, \quad \forall x \in \mathcal{F},\ \forall f \in \mathcal{C}. \tag{7}$$

In addition we impose the following condition on the set of generators $\mathcal{M} \subset \mathcal{U}$:

$$\Delta(\mathcal{M}) \subset \mathrm{span}_{\mathbb{C}}(\mathcal{M}_1 \otimes \mathcal{M}_1) \quad \text{where} \quad \mathcal{M}_1 := \mathcal{M} \cup \{1\}. \tag{8}$$

Then it is natural to define a comultiplication $\tilde{\Delta}$ on $\mathcal{F}$ by the equality $\tilde{\Delta}(x_1 \ldots x_n) := \Delta(x_1) \ldots \Delta(x_n)$, $x_i \in \mathcal{M}$. As $\mathcal{U}$ is a bialgebra $\langle \mathcal{R} \rangle$ must be, at the same time, a coideal.

It is not difficult to check that $\mathcal{F}$ becomes this way a bialgebra and that the triple $(\mathcal{F}, \tilde{\xi}, \mathcal{C})$ fulfills the original conditions (1) and (2), just replacing $\mathcal{U}$ with $\mathcal{F}$ and ξ with $\tilde{\xi}$. One finds that to any mapping $\varphi : \mathcal{M} \to \mathcal{C}$ there exists a unique linear extension $\tilde{\varphi} : \mathcal{F} \to \mathcal{C}$ such that $\tilde{\varphi}(1) = 1$ and the property

$$\tilde{\varphi}(xy) = (\tilde{\xi}_{x_{(1)}} \cdot \tilde{\varphi}(y))\tilde{\varphi}(x_{(2)}), \tag{9}$$

is satisfied for all $x, y \in \mathcal{F}$.

The final step in the construction is to decide when the mapping $\tilde{\varphi}$ can be factorized from $\mathcal{F}$ to $\mathcal{U} = \mathcal{F}/\langle \mathcal{R} \rangle$.

Proposition 2 *Suppose that there is given a mapping $\varphi : \mathcal{M} \to \mathcal{C}$ and let $\tilde{\varphi}$ be its extension to $\mathcal{F}$ as described above. If*

$$(\pi \otimes \tilde{\varphi}) \circ \tilde{\Delta}(\mathcal{R}) = 0 \tag{10}$$

then $\tilde{\varphi}(\langle \mathcal{R} \rangle) = 0$ and so there exists a unique linear mapping $\varphi' : \mathcal{U} \to \mathcal{C}$ such that $\tilde{\varphi} = \varphi' \circ \pi$. Moreover, $\varphi' = 1$ and φ' satisfies the condition (3).

The same conclusions hold true provided $\mathcal{R}$ fulfills a stronger condition than that of being a coideal, namely

$$\tilde{\Delta}(\mathcal{R}) \subset \langle \mathcal{R} \rangle \otimes \mathcal{F} + \mathcal{F} \otimes \mathcal{F}\mathcal{R}, \tag{11}$$

and $\tilde{\varphi}$ satisfies a weaker condition

$$\tilde{\varphi}(\mathcal{R}) = 0. \tag{12}$$

Particularly this construction goes through for the standard deformed enveloping algebras $\mathcal{U} = \mathcal{U}_q(\mathfrak{g})$ in the FRT description [12] where $\mathfrak{g}$ is any simple complex Lie algebra from the four principal series A_ℓ, B_ℓ, C_ℓ and D_ℓ. So the generators are arranged in respectively upper and lower triangular matrices L^+ and L^-, and the set $\mathcal{R}$ is given by the usual RLL relations.

On the other hand the unital algebra $\mathcal{C}$ is generated by quantum anti-holomorphic coordinate functions z^*_{jk}, $j < k$, on the generic dressing orbit of dimension $(\dim_{\mathbb{C}} \mathfrak{g} - \operatorname{rank} \mathfrak{g})/2$. The elements are arranged in an upper triangular matrix Z with units on the diagonal, and the defining relations are given in terms of its Hermitian adjoint Z^*, namely

$$R_{12} Z^*_2 Q Z^*_1 Q^{-1} = Z^*_1 Q Z^*_2 Q^{-1} R_{12} \tag{13}$$

where Q is the diagonal part of the R-matrix R.

The infinitesimal dressing transformation ξ is prescribed on the generators,

$$\xi(L_1^+)\cdot Z_2^* = R_{21}^{-1} Z_2^* Q, \quad \xi(L_1^-)\cdot Z_2^* = Z_1^* Q Z_2^* Q^{-1} (Z_1^*)^{-1}. \tag{14}$$

It can be extended to an arbitrary element from $\mathcal{C}$ with the aid of Leibniz rule (2). The mapping φ is defined on the generators as well,

$$\varphi(L^+) = D^{-1}, \quad \varphi(L^-) = Z^* D^2 (Z^*)^{-1} D^{-1} \tag{15}$$

where D is an arbitrary complex diagonal matrix obeying the conditions

$$\det(D) = 1 \quad \text{and} \quad K_{12} D_1 D_2 = K_{12}. \tag{16}$$

Here K is a matrix related to the R-matrix via the equality

$$R_{12} - R_{21}^{-1} = (q - q^{-1})(P - K_{12}), \tag{17}$$

P stands for the flip operator.

3 Example: $\mathcal{U}_q(\mathfrak{so}(5))$

We shall use the Drinfeld–Jimbo description of $\mathcal{U}_q(\mathfrak{so}(5))$ [13, 14], with the six generators $q^{H_1},\ q^{H_2},\ X_1^+,\ X_2^+,\ X_1^-,\ X_2^-$, the relations

$$\begin{gathered}
[\, q^{H_1},\, q^{H_2}\,] = 0, \\
q^{H_1}\, X_1{}^{\pm} = q^{\pm 1}\, q^{H_1}\, X_1{}^{\pm},\ q^{H_1}\, X_2{}^{\pm} = q^{\mp 1}\, q^{H_1}\, X_2{}^{\pm}, \\
q^{H_2}\, X_1{}^{\pm} = q^{\mp 1}\, q^{H_2}\, X_1{}^{\pm},\ q^{H_2}\, X_2{}^{\pm} = q^{\pm 2}\, q^{H_2}\, X_2{}^{\pm}, \\
[\, X_1{}^+,\, X_1{}^-\,] = \tfrac{q^{H_1} - q^{-H_1}}{q - q^{-1}},\ [\, X_2{}^+,\, X_2{}^-\,] = \tfrac{q^{H_2} - q^{-H_2}}{q - q^{-1}}, \\
[\, X_1^+,\, X_2^-\,] = 0,\ [\, X_2^+,\, X_1^-\,] = 0, \\
(X_2{}^{\pm})^2\, X_1{}^{\pm} - (q^{-1} + q)\ X_2{}^{\pm}\, X_1{}^{\pm}\, X_2{}^{\pm} + X_1{}^{\pm}\, (X_2{}^{\pm})^2 \ = 0, \\
(X_1{}^{\pm})^3\, X_2{}^{\pm} - (q^{-1} + 1 + q)\ (X_1{}^{\pm})^2\, X_2{}^{\pm}\, X_1{}^{\pm} \\
+ (q^{-1} + 1 + q)\ X_1{}^{\pm}\, X_2{}^{\pm}\, (X_1{}^{\pm})^2 - X_2{}^{\pm}\, (X_1{}^{\pm})^3 \ = 0\,.
\end{gathered} \tag{18}$$

and the comultiplication

$$\Delta(q^{H_i}) = q^{H_i} \otimes q^{H_i},\ \Delta(X_i^{\pm}) = X_i^{\pm} \otimes q^{-\frac{1}{2}\, H_i} + q^{\frac{1}{2}\, H_i} \otimes X_i^{\pm}, \quad i = 1, 2. \tag{19}$$

One can pass from the FRT description to the Drinfeld–Jimbo generators using the equalities

$$\begin{gathered}
L_{11}^+ = L_{55}^- = q^{H_1 + H_2},\ L_{22}^+ = L_{44}^- = q^{H_1}, \\
L_{12}^+ = (q - q^{-1})\, q^{-1/2}\, X_2^-\, q^{H_1 + \frac{1}{2} H_2},\ L_{23}^+ = (q - q^{-1})\, q^{-1/2}\, X_1^-\, q^{\frac{1}{2} H_1}, \\
L_{34}^- = (q - q^{-1})\, q^{1/2}\, X_1^+\, q^{\frac{1}{2} H_1},\ L_{45}^- = (q - q^{-1})\, q^{1/2}\, X_2^+\, q^{H_1 + \frac{1}{2} H_2}.
\end{gathered} \tag{20}$$

Only 4 among the 10 generators z^*_{jk}, $1 \le j < k \le 5$, are independent. We denote the independent generators by $w_1, \ldots, w_4$ and make the following choice:

$$z^*_{12} = w_1,\ z^*_{13} = w_2,\ z^*_{14} = w_3,\ z^*_{23} = w_4. \tag{21}$$

The remaining entries can be expressed in terms of $w_1, \ldots, w_4$ as well,

$$\begin{gathered} z^*_{45} = -w_1,\ z^*_{34} = -q^{1/2}\, w_4,\ z^*_{35} = -q^{-1/2}\, w_2 + q^{1/2}\, w_1 w_4, \\ z^*_{24} = -\tfrac{q^{1/2}}{1+q} w_4^2,\ z^*_{15} = -w_1 w_3 - \tfrac{q^{-1/2}}{1+q} w_2^2, \\ z^*_{25} = -q^{-1}\, w_3 - q^{-1/2}\, w_2 w_4 + \tfrac{q^{1/2}}{1+q} w_1 w_4^2\,. \end{gathered} \tag{22}$$

The algebra $\mathcal{C}$ is then determined by the relations

$$\begin{gathered} w_2 w_1 = q\, w_1 w_2,\ w_3 w_2 = q\, w_2 w_3,\ w_3 w_4 = q\, w_4 w_3, \\ w_3 w_1 = w_1 w_3 - q^{-1/2}\,(q-1)\, w_2^2,\ w_4 w_2 = w_2 w_4 - q^{-1/2}\,(q - q^{-1})\, w_3, \\ w_4 w_1 = q^{-1}\, w_1 w_4 + (1 - q^{-2})\, w_2. \end{gathered} \tag{23}$$

Consequently, the ordered monomials $w_1^{n_1} w_2^{n_2} w_3^{n_3} w_4^{n_4}$, $n_1, n_2, n_3, n_4 \in \mathbb{Z}_+$, form an algebraic basis of $\mathcal{C}$.

The infinitesimal dressing transformation is prescribed on the generators as follows:

$$\begin{aligned} &\xi(q^{H_1}) \cdot \{w_1, w_2, w_3, w_4\} = \{q^{-1}\, w_1,\ w_2,\ q\, w_3,\ q\, w_4\}, \\ &\xi(q^{H_2}) \cdot \{w_1, w_2, w_3, w_4\} = \{q^2\, w_1,\ q\, w_2,\ w_3,\ q^{-1}\, w_4\}, \\ &\xi(X_1^-) \cdot \{w_1, w_2, w_3, w_4\} = \{0,\ -q^{1/2}\, w_2,\ q^{1/2}\, w_3,\ -1\}, \\ &\xi(X_2^-) \cdot \{w_1, w_2, w_3, w_4\} = \{-q^{1/2},\ 0,\ 0,\ 0\}, \\ &\xi(X_1^+) \cdot \{w_1, w_2, w_3, w_4\} = \left\{-q^{-1/2}\, w_2,\ q^{-1/2}\, w_3,\ 0,\ q^{1/2}\, w_3,\ \frac{q^{1/2}}{1+q}\, w_4^2\right\}, \\ &\xi(X_2^+) \cdot \{w_1, w_2, w_3, w_4\} \\ &\qquad\qquad = \left\{q^{-1/2}\, w_1^2,\ w_1 w_2,\ -\frac{1}{1+q}\, w_2^2,\ -q^{-1}\, w_1 w_4 + q^{-2}\, w_2\right\}. \end{aligned} \tag{24}$$

Let us turn to the mapping φ. The constraints (16) imply that

$$D = \mathrm{diag}\left(q^{\frac{1}{2}\sigma_1 + \sigma_2}, q^{\frac{1}{2}\sigma_1}, 1, q^{-\frac{1}{2}\sigma_1}, q^{-\frac{1}{2}\sigma_1 - \sigma_2}\right) \tag{25}$$

where $\sigma_1, \sigma_2 \in \mathbb{C}$ are parameters. A straightforward calculation gives

$$\begin{gathered} \varphi(q^{H_1}) = q^{-\frac{1}{2}\sigma_1},\ \varphi(q^{H_2}) = q^{-\sigma_2},\ \varphi(X_1^-) = \varphi(X_2^-) = 0, \\ \varphi(X_1^+) = -\tfrac{q^{\frac{1}{2} - \frac{1}{4}\sigma_1}}{1+q}\, [\sigma_1]_{q^{1/2}}\, w_4,\ \varphi(X_2^+) = -q^{-\frac{1}{2}(1+\sigma_2)}\, [\sigma_2]_q\, w_1, \end{gathered} \tag{26}$$

where $[m]_p := (p^m - p^{-m})/(p - p^{-1})$.

The final step is to calculate the modified action according to the prescription (4). Here is the result:

$$
\begin{aligned}
& q^{H_1} \cdot w_1^{n_1} w_2^{n_2} w_3^{n_3} w_4^{n_4} = q^{-n_1+n_3+n_4-\frac{1}{2}\sigma_1}\, w_1^{n_1} w_2^{n_2} w_3^{n_3} w_4^{n_4}, \\
& q^{H_2} \cdot w_1^{n_1} w_2^{n_2} w_3^{n_3} w_4^{n_4} = q^{2\,n_1+n_2-n_4-\sigma_2}\, w_1^{n_1} w_2^{n_2} w_3^{n_3} w_4^{n_4}, \\
& X_1^- \cdot w_1^{n_1} w_2^{n_2} w_3^{n_3} w_4^{n_4} = \\
& \qquad - q^{\frac{1}{2}(-n_1+n_2-n_3-n_4)+\frac{1}{4}\sigma_1}\, [n_2]_{q^{1/2}}\, w_1^{n_1+1} w_2^{n_2-1} w_3^{n_3} w_4^{n_4} \\
& \qquad + q^{\frac{1}{2}(-n_1+n_3-n_4)+\frac{1}{4}\sigma_1}\, [n_3]_q\, w_1^{n_1} w_2^{n_2+1} w_3^{n_3-1} w_4^{n_4} \\
& \qquad - q^{\frac{1}{2}(-n_1+n_3)+\frac{1}{4}\sigma_1}\, [n_4]_{q^{1/2}}\, w_1^{n_1} w_2^{n_2} w_3^{n_3} w_4^{n_4-1}, \\
& X_2^- \cdot w_1^{n_1} w_2^{n_2} w_3^{n_3} w_4^{n_4} = \\
& \qquad - q^{\frac{1}{2}(1-n_2+n_4+\sigma_2)}\, [n_1]_q\, w_1^{n_1-1} w_2^{n_2} w_3^{n_3} w_4^{n_4}, \\
& X_1^+ \cdot w_1^{n_1} w_2^{n_2} w_3^{n_3} w_4^{n_4} = \\
& \qquad - q^{-1+\frac{1}{2}(n_1-n_3-n_4)+\frac{1}{4}\sigma_1}\, [n_1]_q\, w_1^{n_1-1} w_2^{n_2+1} w_3^{n_3} w_4^{n_4} \\
& \qquad + q^{-1+\frac{1}{2}(-n_1+n_2-n_3-n_4)+\frac{1}{4}\sigma_1}\, [n_2]_{q^{1/2}}\, w_1^{n_1} w_2^{n_2-1} w_3^{n_3+1} w_4^{n_4} \\
& \qquad + \frac{q^{\frac{1}{2}(1-n_1+n_3)-\frac{1}{4}\sigma_1}}{1+q}\, [n_4-\sigma_1]_{q^{1/2}}\, w_1^{n_1} w_2^{n_2} w_3^{n_3} w_4^{n_4+1}, \\
& X_2^+ \cdot w_1^{n_1} w_2^{n_2} w_3^{n_3} w_4^{n_4} = \\
& \qquad q^{-\frac{1}{2}(1-n_2+n_4+\sigma_2)}\, [n_1+n_2-n_4-\sigma_2]_q\, w_1^{n_1+1} w_2^{n_2} w_3^{n_3} w_4^{n_4} \\
& \qquad - \frac{q^{-1+n_1+\frac{1}{2}n_2+n_3-\frac{3}{2}(n_4+\sigma_2)}}{1+q}\, [n_3]_q\, w_1^{n_1} w_2^{n_2+2} w_3^{n_3-1} w_4^{n_4} \\
& \qquad + q^{-1+n_1+\frac{1}{2}n_2+n_3-n_4-\frac{3}{2}\sigma_2}\, [n_4]_{q^{1/2}}\, w_1^{n_1} w_2^{n_2+1} w_3^{n_3} w_4^{n_4-1} \\
& \qquad - (q-1)\, q^{-\frac{5}{2}+n_1+\frac{1}{2}(n_2-n_4)-\frac{3}{2}\sigma_2}\, [n_4]_{q^{1/2}} [n_4-1]_{q^{1/2}} \\
& \qquad\qquad \times\ w_1^{n_1} w_2^{n_2} w_3^{n_3+1} w_4^{n_4-2}.
\end{aligned}
\tag{27}
$$

Note that $1 \in \mathcal{C}$ is a lowest weight vector ($X_1^- \cdot 1 = X_2^- \cdot 1 = 0$), with the lowest weight determined by $q^{H_1} \cdot 1 = q^{-\frac{1}{2}\sigma_1}$, $q^{H_2} \cdot 1 = q^{-\sigma_2}$. Consequently, the cyclic submodule $\mathcal{U} \cdot 1$ is finite-dimensional and irreducible provided $\sigma_1, \sigma_2 \in \mathbb{Z}_+$, and this way one can obtain, in principle, all finite-dimensional irreducible representations of $\mathcal{U}_q(\mathfrak{so}(5))$. For example, if $\sigma_1 = 1$, $\sigma_2 = 0$, then $\mathcal{U} \cdot 1$ is a 4-dimensional vector space spanned by the vectors: 1, w_4, $w_2 - q\, w_1 w_4$, $(1+q) w_3 + q^{3/2}\, w_2 w_4$.

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References

[1] Šťovíček P.: J. Math. Phys. *34* (1993) 1606.

[2] Šťovíček P.: Commun. Math. Phys. *158* (1993) 135.

[3] Jurčo B., and Šťovíček P.: Commun. Math. Phys. *182* (1996) 221.

[4] Šťovíček P.: J. Algebra *184* (1996) 71.

[5] Šťovíček, P., and Twarock, R.: J. Math. Phys. *38* (1997) 1161.

[6] Sazdjian H., Stanev Y. S., and Todorov I. T.: J. Math. Phys. *36* (1995) 2030.

[7] Jurčo B., and Schlieker, M.: J. Math. Phys. *36* (1995) 3814.

[8] Parshall B., and Wang J.: Quantum linear groups (Rhode Island, AMS 1991).

[9] E. Taft E., and Towber J.: J. Algebra *142* (1991) 1.

[10] Biedenharn L. C., and Lohe M. H: Commun. Math. Phys. *146* (1992) 483.

[11] Dabrowski L., Dobrev V. K., and Floreani R.; J. Math. Phys. *35* (1994) 971.

[12] Faddeev L. D., Reshetikhin N. Yu., and Takhtajan L. A.: Leningrad Math. J. *1* (1990) 193.

[13] Drinfeld V. G.: Proc. ICM Berkley 1986 (AMS 1987) 798.

[14] Jimbo M.: Lett. Math. Phys. *10* (1985) 63.